

# On distributed monitoring of asynchronous systems

Volker Diekert<sup>1</sup> and Anca Muscholl<sup>2</sup>

<sup>1</sup> Universität Stuttgart, FMI, Germany

<sup>2</sup> LaBRI, Univ. of Bordeaux, France

## 1 Introduction

Distributed systems are notoriously difficult to understand and analyze in order to assert their correction w.r.t. given properties. They often exhibit a huge number of different behaviors, as soon as the active entities (peers, agents, processes, ...) behave in an asynchronous manner. Already the modelization of such systems is a non-trivial task, let alone their formal verification.

Several automata-based distributed models have been proposed and studied over the past twenty years, capturing various aspects of distributed behavior. Depending on the motivation, such models fall into two large categories. In the first one we find rather simple models, expressing basic synchronization mechanisms, like Petri nets or communicating automata. In the second category we see more sophisticated models, conceived for supporting practical system design, like statecharts or I/O automata. It is clear that being able to develop automated verification techniques requires a good understanding of the simpler models, in particular since more complex ones are often built as a combination of basic models.

The purpose of this paper is to discuss the problem of distributed monitoring on a simple model of finite-state distributed automata based on shared actions, called *asynchronous automata*. Monitoring is a question related to runtime verification: assume that we have to check a property  $L$  against an unknown or very complex system  $\mathcal{A}$ , so that classical static analysis is not possible. Therefore instead of model-checking a *monitor* is used, that checks the property on the underlying system at runtime. The question is which properties can be checked in this way, that is, which properties  $L$  are *monitorable*. A classical example for monitorable properties are safety properties, like “no alarm is raised”. A monitor for a property  $L$  is an automaton  $\mathcal{M}_L$  that after each finite execution tells whether (1) every possible extension of the execution is in  $L$ , or (2) every possible extension is in the complement of  $L$ , or neither (1) nor (2) holds. The notion of monitorable properties has been proposed by Pnueli and Zaks [15], and the theory has been extended to various kinds of systems, for instance to probabilistic systems [3,10] or real-time systems [1,2].

We are interested here in monitoring distributed systems modelled as asynchronous automata. It is natural to require that monitors should be of the same kind as the underlying system, so we consider here distributed monitoring. A distributed monitor does not have a global view of the system, therefore we propose

the notion of *locally monitorable* trace language. Our main result shows that if the distributed alphabet of actions is connected and if  $L$  is a set of  $\Gamma$ -infinite traces (for some subset of processes  $\Gamma$ ) such that both  $L$  and its complement  $L^c$  are countable unions of locally safety languages, then  $L$  is locally monitorable. We also show that over  $\Gamma$ -infinite traces, recognizable countable unions of locally safety languages are precisely the complements of deterministic languages.

## 2 Preliminaries

The idea of describing concurrency by a fixed independence relation on a given set of actions  $SS$  goes back to the late seventies, to Mazurkiewicz [12] and Keller [11] (see also [6]). One can start with a *distributed action alphabet*  $(SS, dom)$  on a finite set  $Proc$  of processes, where  $dom : SS \rightarrow (2^{Proc} \setminus \emptyset)$  is a *location function*. The location  $dom(a)$  of action  $a \in SS$  comprises all processes that need to synchronize in order to perform this action. It defines in a natural way an *independence relation*  $I \subseteq SS \times SS$  by letting  $(a, b) \in I$  if and only if  $dom(a) \cap dom(b) = \emptyset$ .

The execution order of two independent actions  $(a, b) \in I$  is irrelevant, they can be executed as  $a, b$ , or  $b, a$  - or even concurrently. More generally, we can consider the congruence  $\sim_I$  on  $SS^*$  generated by  $I$ . An equivalence class  $[w]_I$  of  $\sim_I$  is called a (finite) Mazurkiewicz *trace*, and it can be also viewed as labeled pomset  $t = \langle V, \leq, \lambda \rangle$  of a special kind: if  $w = a_0 \cdots a_n$  then the vertex set is  $V = \{0, \dots, n\}$ , the labeling function is  $\lambda(i) = a_i$  and  $\leq = (\{(i, j) \mid i < j, (a_i, a_j) \notin I\})^*$  is the partial order. The word  $w$  is a *linearization* of  $t$  defined as above, i.e., a total order compatible with the partial order of  $t$ .

*Infinite traces* can be defined in a similar way from  $\omega$ -words. Finite and infinite traces are also called *real traces*, and the set of real traces is written  $\mathbb{R}(SS, I)$  (or simply  $\mathbb{R}$  when  $SS, I$  are clear from the context). A trace  $t$  is a *prefix* of a trace  $t'$  (denotes as  $t \leq t'$ ) if  $t$  is isomorphic to a downwards-closed subset of  $t'$ . The set of prefixes of  $t$  is denoted  $pref(t)$ . If  $L \subseteq \mathbb{R}$  then we denote by  $Lin(L) \subseteq SS^\infty$  the set of linearizations of traces from  $L$ .

A language  $K \subseteq SS^\infty$  is called *trace-closed* if  $K = Lin(L)$  for some  $L \subseteq \mathbb{R}$ . Whenever convenient, we talk about trace languages  $L \subseteq \mathbb{R}$  or trace-closed word languages  $K \subseteq SS^\infty$  in equivalent terms. A language  $L \subseteq \mathbb{R}$  is *recognizable* if  $Lin(L) \subseteq SS^\infty$  is a regular language of finite and infinite words.

Linear temporal properties like *safety* and *liveness* [14] can be translated into topological properties, as closed and dense sets in the Cantor topology. For real traces, these notions generalize smoothly to the Scott topology, by replacing word prefixes by trace prefixes. The Scott topology corresponds to a global view in traces, where one needs to reason on global configurations, i.e., configurations involving several processes. However, in the setting of monitoring that we discuss here, such a global view is not available. Therefore we use here *local safety* as basic notion, as introduced in [4] and explained in the following.

A trace  $t = \langle V, \leq, \lambda \rangle$  is called *prime* if it is finite and has a unique maximal element. That is,  $|\max(t)| = 1$ , where  $\max(t)$  is the set of maximal elements of

$t$  w.r.t. the partial order  $\leq$ . The set of prime traces in  $\mathbb{R}$  is denoted  $\mathbb{P}(\mathbb{R})$ . The set of prime prefixes of elements of  $L \subseteq \mathbb{R}$  is denoted  $\mathbb{P}(L)$ .

**Definition 1.** Let  $L \subseteq \mathbb{R}$ .

1.  $L$  is called *prime-open* if it is of the form  $\bigcup \{p\mathbb{R} \mid p \in U\}$  for some  $U \subseteq \mathbb{P}$ . Complements of prime-open sets are called *prime-closed*.
2.  $\bar{L}$  is the intersection of all prime-closed sets containing  $L$  (and denoted as *prime-closure* of  $L$ ). Note that  $\bar{L}$  is prime-closed.
3. A prime-closed, recognizable language  $L \subseteq \mathbb{R}$  is called a *locally safety language*.

*Remark 1.* 1. Every prime-open set is also Scott-open, and prime-open sets are closed under union, but not under intersection. As an example consider  $a\mathbb{R} \cap b\mathbb{R}$  which is not prime-open for  $(a, b) \in I$ .  
2. A first-order locally safety language  $L \subseteq \mathbb{R}$  is a prime-closed set such that  $\text{Lin}(L)$  is a first-order language. It is known from [4] that first-order locally safety languages are characterized by formulas of the form  $G\psi$ , with  $\psi$  a past formula in a local variant of LTL called LocTL.

We end this section by introducing our model for distributed automata. An *asynchronous automaton*  $\mathcal{A} = \langle (S_\alpha)_{\alpha \in \text{Proc}}, s_{in}, (\delta_a)_{a \in SS} \rangle$  is given by

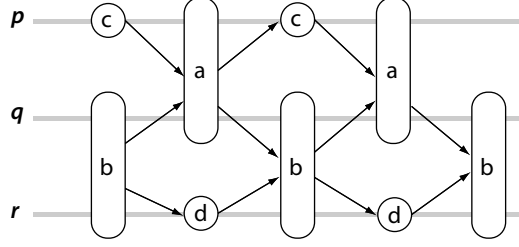
- for every process  $\alpha$  a finite set  $S_\alpha$  of (local) states,
- the initial state  $s_{in} \in \prod_{\alpha \in \text{Proc}} S_\alpha$ ,
- for every action  $a \in SS$  a transition relation  $\delta_a \subseteq (\prod_{\alpha \in \text{dom}(a)} S_\alpha)^2$  on tuples of states of processes in  $\text{dom}(a)$ .

For convenience, we abbreviate a tuple  $(s_\alpha)_{\alpha \in P}$  of local states by  $s_P$ , where  $P \subseteq \text{Proc}$ . We also denote  $\prod_{\alpha \in \text{Proc}} S_\alpha$  as *global states* and  $\prod_{\alpha \in P} S_\alpha$  as  $S_P$ .

An asynchronous automaton can be seen as a sequential automaton with the state set  $S = \prod_{\alpha \in \text{Proc}} S_\alpha$  and transitions  $s \xrightarrow{a} s'$  if  $(s_{\text{dom}(a)}, s'_{\text{dom}(a)}) \in \delta_a$ , and  $s_{\text{Proc} \setminus \text{dom}(a)} = s'_{\text{Proc} \setminus \text{dom}(a)}$ . By  $\mathcal{L}(\mathcal{A})$  we denote the set of words labeling runs of this sequential automaton that start from the initial state. It can be easily noted that  $\mathcal{L}(\mathcal{A})$  is trace-closed. The automaton is *deterministic* if each  $\delta_a$  is a (partial) function.

*Example 1.* Let us consider the asynchronous automaton  $\mathcal{A}$  given by  $S_p = \{0\}$ ,  $S_q = S_r = \{0, 1\}$ , and transition function  $\delta_a(s_p, s_q) = (s_p, \neg s_q)$  if  $s_q = 1$  (undefined otherwise),  $\delta_d(s_r) = \neg s_r$  if  $s_r = 1$  (undefined otherwise),  $\delta_b(s_q, s_r) = (1, 1)$  if  $s_q \wedge s_r = 0$  (undefined otherwise) and  $\delta_c(s_p) = s_p$ . Starting with  $s_0 = (0, 0, 0)$ , an accepting run of  $\mathcal{A}$  checks that between any two successive  $b$ -events, there is either an  $a$  or a  $d$  (or both), and there is a  $b$ -event before all  $a$  and  $d$ .

Since the notion of a trace was formulated without a reference to an accepting device, it is natural to ask if the model of asynchronous automata is powerful enough for capturing the notion of regularity. Zielonka's theorem below says that this is indeed the case, hence these automata are a right model for the simple view of concurrency captured by Mazurkiewicz traces.



**Fig. 1.** The pomset associated with the trace  $t = [cbadcbadb]$ , with  $\text{dom}(a) = \{p, q\}$ ,  $\text{dom}(b) = \{q, r\}$ ,  $\text{dom}(c) = \{p\}$ ,  $\text{dom}(d) = \{r\}$ .

**Theorem 1.** [17] Let  $\text{dom} : SS \rightarrow (2^{\text{Proc}} \setminus \{\emptyset\})$  be a distribution of letters. If a language  $L \subseteq SS^*$  is regular and trace-closed then there is a deterministic asynchronous automaton accepting  $L$  (of size exponential in the number of processes and polynomial in the size of the minimal automaton for  $L$ , see [9]).

### 3 Safety languages

A set of traces  $C \subseteq \mathbb{R}$  is called *coherent* if  $C \subseteq \text{pref}(t)$  for some  $t \in \mathbb{R}$ . This means that  $\sqcup C \in \mathbb{R}$  exists, and it is a prefix of  $t$ . By  $L^c$  we denote the complement  $\mathbb{R} \setminus L$  of  $L$ . Recall that  $\mathbb{P}(L)$  is the set of prime prefixes of traces in  $L \subseteq \mathbb{R}$ .

We use in our characterizations below a basic property of automata on traces, which is for instance satisfied by (runs of) asynchronous automata, called *forward diamond property*. A set  $K \subseteq SS^*$  satisfies the forward diamond property if the following holds:

If  $ua \in K$  and  $ub \in K$ , then  $uab \in K$ , for every  $u \in SS^*$  and  $(a, b) \in I$ .

**Lemma 1.** For  $L \subseteq \mathbb{R}$  we have

$$\overline{L} = \{\sqcup C \mid C \subseteq \mathbb{P}(L) \text{ and } C \text{ is coherent}\}.$$

We have  $\overline{L} = \overline{K}$  if and only if  $\mathbb{P}(L) = \mathbb{P}(K)$ .

*Proof.* Let  $X = \{\sqcup C \mid C \subseteq \mathbb{P}(L) \text{ and } C \text{ is coherent}\}$ . By definition,  $X^c = U\mathbb{R}$  with  $U = \mathbb{P} \setminus \mathbb{P}(L)$ , thus  $X$  is prime-closed (and contains  $L$ ). Let  $K \supseteq L$  be prime-closed, thus  $K^c = V\mathbb{R}$  with  $V \subseteq \mathbb{P}$ . Consider some coherent set  $C \subseteq \mathbb{P}(L)$ , and assume that  $\sqcup C \in v\mathbb{R}$  for some  $v \in V$ . But then  $v \in \mathbb{P}(L)$ , thus  $K^c \cap L \neq \emptyset$ , a contradiction. So  $X \subseteq K$ , which shows that  $\overline{L} = X$ .

**Lemma 2.** If  $L \subseteq \mathbb{R}$  is recognizable, then the prime closure  $\overline{L}$  is recognizable, too. Moreover, on input  $(SS, \text{dom})$  and (sequential) Büchi automaton  $\mathcal{B}$  such that  $L = \mathcal{L}(\mathcal{B})$  is trace-closed, we can compute an exponential-size, deterministic asynchronous automaton  $\mathcal{A}$  accepting  $\overline{L}$ , such that all states of  $\mathcal{A}$  are final.

*Proof.* Given  $L \subseteq \mathbb{R}$  recognizable, we have that  $\mathbb{P}(L)$  is recognizable, too. Then it is easy to see that  $\bar{L}$  is recognizable, by using for instance monadic second-order logic over traces.

Let us consider the complexity of the construction of a deterministic asynchronous automaton for  $\bar{L}$  in more detail. We assume that the input  $L$  is given by a (sequential) Büchi automaton  $\mathcal{B}$ . We first determinize  $\mathcal{B}$  and get a deterministic (say Rabin) automaton  $\mathcal{B}'$  for  $L$ . From  $\mathcal{B}'$  we can easily construct a DFA accepting  $\mathbb{P}(L)$ : we just need to store the set of maximal processes in the control state. The resulting DFA is exponential in both  $\mathcal{B}$  and  $Proc$ . By applying the construction cited in Thm. 1 we obtain a deterministic asynchronous automaton  $\mathcal{A}$  for  $\mathbb{P}(L)$  which is still exponential in  $\mathcal{B}$  and  $Proc$ . Using classical timestamping we may assume that each local state reached by the maximal processes of a prime trace contains the complete information about the global state of  $\mathcal{A}$  reached on that prime trace - the size of the deterministic asynchronous automaton  $\mathcal{A}'$  thus obtained remains exponential. It remains to construct the automaton accepting  $\bar{L}$ . Recall that  $\bar{L}$  contains precisely those traces where all prime prefixes belong to  $\mathbb{P}(L)$ . Thus, it suffices to take  $\mathcal{A}'$  and forbid transitions that produce bad local states of  $\mathcal{A}'$ , that is, local states that are non-final viewed as global states of  $\mathcal{A}$ . On finite or infinite traces, the automaton  $\mathcal{A}'$  accepts precisely  $\bar{L}$ . By construction, all its reachable states are final.

**Proposition 1.** *The following are equivalent characterizations for  $L \subseteq \mathbb{R}$ :*

1.  $L$  is a locally-safety language.
2.  $K = Lin(L) \subseteq SS^\infty$  is a regular, prefix-closed language such that  $K \cap SS^\omega$  is a safety language, and  $K \cap SS^*$  satisfies the forward diamond condition.
3.  $L$  is accepted by a deterministic asynchronous automaton where all reachable states are final.

*Proof.* The implications (1)  $\Rightarrow$  (2) and (3)  $\Rightarrow$  (1) are immediate. For (2)  $\Rightarrow$  (3) let us assume that  $K = Lin(L)$  is regular, prefix-closed and satisfies the two additional conditions in the statement. Since  $K \cap SS^*$  is prefix-closed, trace-closed and satisfies the forward diamond property, there exists a deterministic asynchronous automaton  $\mathcal{B}$  recognizing  $K \cap SS^*$  (equivalently, the set of finite traces in  $L$ ) such that all reachable states are final [16]. Since  $K$  is assumed to be prefix-closed and  $K \cap SS^\omega$  is a safety language, we obtain that the automaton  $\mathcal{B}$  accepts precisely  $\bar{L} = L$  over  $\mathbb{R}$ .

*Example 2.* Assume that  $SS = \{a, b, c\}$  with  $dom(a) = \{\alpha\}$ ,  $dom(b) = \{\beta\}$  and  $dom(c) = \{\alpha, \beta\}$ . The trace language “no two consecutive  $c$ ’s” is a locally safety language, and it can be recognized by an asynchronous automaton where both processes remember their last action, and do not allow two consecutive  $c$ ’s.

The trace language “no  $a$  in parallel with a  $b$ ” is not a locally safety language (but it is Scott-closed).

For first-order languages we have, as usual, also a characterization by temporal logics:

**Proposition 2.** *The following are equivalent characterizations for  $L \subseteq \mathbb{R}$ :*

1.  *$L$  is a locally-safety language definable in first-order logic.*
2.  *$L$  is definable by a globally past formula in LocTL.*
3.  *$K = \text{Lin}(L) \subseteq SS^\infty$  is a first-order, prefix-closed language such that  $K \cap SS^\omega$  is a safety language, and  $K \cap SS^*$  satisfies the forward diamond property.*

*Proof.* The equivalence (1)  $\Leftrightarrow$  (2) follows from [4], and the implication (1)  $\Rightarrow$  (3) is immediate. For (3)  $\Rightarrow$  (1) it suffices to show that  $L = \overline{L}$  (since we know by [7] that  $L$  must be first-order). So let  $t = \sqcup C$ , with  $C \subseteq \mathbb{P}(L)$  coherent. For every  $u \in \mathbb{P}(L)$  and every linearization  $x$  of  $u$ , we have  $x \in K$  since  $K$  is prefix-closed. Moreover, if  $\{t, t'\}$  is coherent and  $K$  contains all linearizations of  $t$  and  $t'$ , respectively, then by the forward diamond property,  $K$  contains some (and thus all) linearization(s) of  $t \sqcup t'$ . This shows the claim for finite traces  $t$ . For infinite traces it follows from  $K \cap SS^\omega$  being a safety language.

## 4 Local monitoring

Here and in the following we write  $s \leq L$  for a (finite) trace  $s \in \mathbb{R}$  and a language  $L \subseteq \mathbb{R}$  if there exists some  $t \in L$  with  $s \leq t$ .

**Definition 2.** *A set  $L \subseteq \mathbb{R}$  is called locally monitorable if for all  $s \in \mathbb{P}$  there exists some  $t \in \mathbb{P}$  with (1)  $s \leq t\mathbb{R}$  and (2) either  $t\mathbb{R} \subseteq L$  or  $t\mathbb{R} \subseteq L^c$ .*

Notice that in the definition of locally monitorable sets, the first condition says that  $\{s, t\}$  is coherent. So a set  $L$  is *locally monitorable* if for every *prime* trace  $s$  there is another *prime* trace  $t$  that is coherent with  $s$  and such that after  $t$  we know that every extension belongs either to  $L$  or to its complement  $L^c$ .

The following lemma extends a well-known observation from words to traces:

**Lemma 3.** *Every prime-closed trace language is locally monitorable. In particular, every locally-safety (or locally-co-safety) language is locally monitorable.*

*Proof.* Let  $L = \overline{L}$  and  $s \in \mathbb{P}$ . If  $s\mathbb{R}$  is not a subset of  $L$ , then there exists some  $t = sx \in L^c$ . Since  $L$  is prime-closed this means that there is some  $u \in \mathbb{P} \setminus \mathbb{P}(L)$  with  $u \leq t$ . But then  $\{u, s\}$  is coherent, thus  $s \leq u\mathbb{R}$  and  $u\mathbb{R} \subseteq L^c$ .

The next proposition characterizes locally monitorable sets in terms of the closure operator defined in the previous section:

**Proposition 3.**  *$L \subseteq \mathbb{R}$  is locally monitorable if and only if  $\overline{L} \cap \overline{L}^c$  does not contain any non-empty prime-open subset.*

*Proof.* First, assume by contradiction that  $L$  is locally monitorable, but  $s\mathbb{R} \subseteq \overline{L} \cap \overline{L}^c$  for some  $s \in \mathbb{P}$ . By symmetry in  $L$  and  $L^c$  we may assume that we find  $t \in \mathbb{P}$  and  $s \leq t\mathbb{R} \subseteq L$ . Hence,  $t \notin \mathbb{P}(L^c)$  and thus  $t\mathbb{R} \cap \overline{L}^c = \emptyset$ . But  $s\mathbb{R} \cap t\mathbb{R} \neq \emptyset$ . Contradiction.

For the other direction let  $s \in \mathbb{P}$ . We may assume (again by symmetry in  $L$  and  $L^c$ ) that  $s\mathbb{R} \cap \overline{L}^c \neq \emptyset$ . Hence, there is  $x \notin \overline{L}$  with  $s \leq x$ . This implies that there is  $t \in \mathbb{P} \setminus \mathbb{P}(L)$  with  $s \leq t\mathbb{R}$ . Thus,  $t\mathbb{R} \subseteq L^c$  and  $L$  is locally monitorable.

We state now the main result of this section, which shows that whenever a recognizable property over traces is locally monitorable, we can build a monitor that is of the same type as the system on which it runs, i.e., an asynchronous automaton.

**Theorem 2.** *Let  $L \subseteq \mathbb{R}$  be recognizable. Then we can decide whether  $L$  is locally monitorable. Moreover, if  $L$  is locally monitorable, then we find a deterministic asynchronous finite state monitor for  $L$ .*

*Proof.* By Lemma 2 there exist deterministic asynchronous automata  $\mathcal{A}$ ,  $\mathcal{A}'$  accepting  $\bar{L}$  and  $\bar{L}^c$ , resp., such that all their reachable states are final.

Let  $(\delta_a)_{a \in SS}, (\delta'_a)_{a \in SS}$  be the transition functions of  $\mathcal{A}, \mathcal{A}'$ , resp. We modify the product automaton  $\mathcal{A} \times \mathcal{A}'$  to a (deterministic) asynchronous automaton  $\mathcal{C}$  with transition functions  $(\Delta_a)_{a \in SS}$ : first we add two local states  $\perp_\alpha, \top_\alpha$  on each process  $\alpha \in Proc$ . Consider  $a \in \Sigma$  and some trace  $t$  on which  $\mathcal{A}$  reaches state  $s$  and  $\mathcal{A}'$  reaches state  $s'$ . Note that  $ta$  belongs to one of  $\bar{L}$  or  $\bar{L}^c$  (or both). If  $\mathcal{A}$  has no  $a$ -transition on  $s_{dom(a)}$  then we add  $\Delta_a((s_\alpha, s'_\alpha)_{\alpha \in dom(a)}) = (\perp_\alpha)_{\alpha \in dom(a)}$ . If  $\mathcal{A}'$  has no  $a$ -transition on  $s'_{dom(a)}$  then we add the transition  $\Delta_a((s_\alpha, s'_\alpha)_{\alpha \in dom(a)}) = (\top_\alpha)_{\alpha \in dom(a)}$ . The first case corresponds to  $ta\mathbb{R} \cap \bar{L} = \emptyset$ , the second one to  $ta\mathbb{R} \cap \bar{L}^c = \emptyset$ . Else,  $\Delta_a((s_\alpha, s'_\alpha)_{\alpha \in dom(a)})$  is defined as the componentwise product of  $\delta_a(s_{dom(a)})$  and  $\delta'_a(s'_{dom(a)})$ . Finally, for each  $a \in SS$  and each tuple  $\hat{s}_{dom(a)}$  of states of  $\mathcal{A} \times \mathcal{A}'$ : if some component of  $\hat{s}_{dom(a)}$  is  $\perp$ , then all components of  $\Delta_a(\hat{s}_{dom(a)})$  become  $\perp$ , and symmetrically for  $\top$ . The language  $L$  is not locally monitorable if and only if the automaton  $\mathcal{C}$  has some infinite run where no process gets into state  $\top$  or  $\perp$ .

**Proposition 4.** *The following problem is PSPACE-hard:*

- *Input:* A Büchi automaton  $\mathcal{B} = \langle Q, \Sigma, \delta, q_0, F \rangle$ .
- *Question:* Is the accepted language  $\mathcal{L}(\mathcal{B}) \subseteq \Sigma^\omega$  monitorable?

*Proof.* The universality problem for non-deterministic finite automata (NFA) is one of the well-known PSPACE complete problems. We reduce this problem to the problem of monitorability.

Start with an NFA  $\mathcal{A} = \langle Q', \Gamma, \delta', q_0, F' \rangle$ . We will construct a Büchi automaton  $\mathcal{B}$  such that we have  $\mathcal{L}(\mathcal{A}) = \Gamma^*$  if and only if  $\mathcal{L}(\mathcal{B}) \subseteq \Sigma^\omega$  is monitorable.

For this we use a new letter  $b$  and we let  $\Sigma = \Gamma \cup \{b\}$ . We use three new states  $d, e, f$  and we let  $Q = Q' \cup \{d, e, f\}$ . The repeated (or final) states of  $\mathcal{B}$  are defined as  $F = \{e, f\}$ . The initial state is the same as before:  $q_0$ . It remains to define  $\delta$ . We keep all arcs from  $\delta'$  and we add the following new arcs.

- $q \xrightarrow{b} d \xrightarrow{a} e \xrightarrow{a} e$  for all  $q \in Q' \setminus F'$  and all  $a \in \Gamma$ .
- $e \xrightarrow{b} d \xrightarrow{b} d$
- $q \xrightarrow{b} f \xrightarrow{c} f$  for all  $q \in F'$  and all  $c \in \Sigma$ .

In order to understand the construction, consider what happens if we reach state  $d$  or state  $f$ . Starting in  $f$  we accept everything, because we loop in a final state

of  $\mathcal{B}$ . On the other hand starting in  $d$  we accept all words except those which end in  $b^\omega$ . Starting in  $d$  we are nowhere monitorable.

Now, let  $w \in \Sigma^*$ . This can be written as  $uv$  where  $u \in \Gamma^*$  is the maximal prefix without any occurrence of  $b$ .

Assume we have  $\mathcal{L}(\mathcal{A}) = \Gamma^*$ , then there is path from  $q_0$  to  $f$  labelled by  $wb$  since reading  $u$  leads us to some state in  $F'$ . This implies that  $wb\Sigma^\omega \subseteq \mathcal{L}(\mathcal{B})$  for all  $w \in \Gamma^*$ ; and  $\mathcal{L}(\mathcal{B})$  is monitorable.

On the other hand, if  $\mathcal{L}(\mathcal{A}) \neq \Gamma^*$ , then there is some word  $u \in \Gamma^*$  such that  $u$  leads to states in  $Q' \setminus F'$ , only. Thus, reading  $ub$  we are necessarily in state  $d$ . The language  $\mathcal{L}(\mathcal{B})$  is not monitorable, due to the word  $ub \in \Sigma^*$ .

We have a matching upper bound for Büchi automata in the theorem below. Note that the input is a Büchi automaton accepting a trace-closed language, therefore we may see the accepted language also as a subset of  $\mathbb{R}$ .

**Theorem 3.** *The following problem is PSPACE-complete:*

- *Input: A Büchi automaton  $\mathcal{B} = \langle Q, \Sigma, \delta, q_0, F \rangle$  and  $(SS, dom)$  such that  $\mathcal{L}(\mathcal{B})$  is trace-closed.*
- *Question: Is the accepted language  $\mathcal{L}(\mathcal{B}) \subseteq \mathbb{R}$  locally monitorable?*

*Proof.* For a subset  $P \subseteq Q$  let us write  $\mathcal{L}(\mathcal{B}, P)$  for the accepted language of  $\mathcal{B}$  when  $P$  is used as a set of initial states. We say that  $P$  is *good* if either  $\mathcal{L}(\mathcal{B}, P) = \Sigma^\omega$  or  $\mathcal{L}(\mathcal{B}, P) = \emptyset$ . The predicate whether  $P$  is good can be computed in PSPACE. For a letter  $a \in \Sigma$  and  $P, P' \subseteq Q$  we define another predicate  $\text{Reach}(P, P', a)$ , which is defined to be *true*, if:

$$P' = \{q \in Q \mid \exists p \in P \exists ta \in \mathbb{P} \text{ and } p \xrightarrow{ta} q\}.$$

Note that  $\text{Reach}(P, P', a)$  is computable in PSPACE, too. If there is no  $a \in \Sigma$  such that  $\text{Reach}(\{q_0\}, P', a)$  becomes true for some good  $P' \subseteq Q$ , then  $L = \mathcal{L}(\mathcal{B})$  is not locally monitorable. Thus, we may assume that such  $P$  and  $a$  exist. If there are two letters  $a$  and  $b$  in different connected components of  $(\Sigma, dom)$  with this property, then  $L$  is locally monitorable. Hence we assume in the following that there is only one component where such a letter  $a$  exist. Indeed, letters occurring in some prime traces belong to a single connected component of  $(\Sigma, dom)$ ; and due to  $\text{Reach}(\{q_0\}, P', a)$  it is enough to consider monitorability of prime traces which belong to the same component as the letter  $a$ . Since every such prime trace can be made longer such that it ends with this letter  $a$ , we fix  $a$  in the following.

Now, the language  $L \subseteq \mathbb{R}$  is locally monitorable if and only if for all  $P \subseteq Q$  such that  $\text{Reach}(\{q_0\}, P, a)$  holds, there is some good subset  $P'$  such that we have  $\text{Reach}(P, P', a)$ .

To see this, let  $L \subseteq \mathbb{R}$  be locally monitorable. Consider a subset  $P$  such that  $\text{Reach}(\{q_0\}, P, a)$  holds. This corresponds to some word  $s$  such that the corresponding trace  $s = s'a$  is a prime. Since  $L$  is locally monitorable, there exists some prime  $t$  such that  $s \leq t\mathbb{R}$  and either  $t\mathbb{R} \subseteq L$  or  $t\mathbb{R} \subseteq L^c$ . However,



by the assumption above, we may assume that  $s$  and  $t$  belong to the same component. We can make  $t$  longer and actually assume  $s \leq t$  and such that  $t = t'a$ . Choose some representing word  $w$  for  $t$ . If  $P'$  is the subset of states we can reach after reading  $w$  starting in  $q_0$  we have  $\text{Reach}(P, P', a)$ . The set  $P'$  is good, because  $L$  is trace-closed. Indeed, if  $t\mathbb{R} \subseteq L$ , then  $w\Sigma^\omega \subseteq L$ , hence  $\mathcal{L}(\mathcal{B}, P') = \Sigma^\omega$ . If  $t\mathbb{R} \subseteq L^c$ , then  $\mathcal{L}(\mathcal{B}, P') = \emptyset$ .

For the converse it is clear that the condition is strong enough to ensure local monitorability of  $L$ .

The condition to monitor a single language might be an unnecessary restriction. We can imagine a certain family of properties or languages  $L_1, \dots, L_n$  and we content ourselves with a monitor which selects one of these possibilities, even if certain  $L_i$  and  $L_j$  do intersect non-trivially for  $i \neq j$ . This leads to the following definition.

**Definition 3.** Let  $n \in \mathbb{N}$  and  $L_1, \dots, L_n$  be subsets of  $\mathbb{R}$ . We say that the family  $\{L_1, \dots, L_n\}$  is locally monitorable, if

$$\forall s \in \mathbb{P} \exists t \in \mathbb{P} \exists 1 \leq i \leq n : s \leq t\mathbb{R} \subseteq L_i.$$

*Remark 2.* A language  $L$  is locally monitorable if and only if the family  $\{L, L^c\}$  is locally monitorable.

A distributed alphabet  $(SS, dom)$  can be split into several *connected components*. This is a partition  $SS = SS_1 \cup \dots \cup SS_k$  such that all  $SS_i$  are non-empty and  $SS_i \times SS_j \subseteq I$  for all  $1 \leq i < j \leq k$ . We say that  $(SS, dom)$  is *connected*, if  $k = 1$  and *disconnected* otherwise. For  $k \geq 2$  we can write  $\mathbb{R} = \mathbb{R}' \times \mathbb{R}''$  such that  $\mathbb{R}'$  and  $\mathbb{R}''$  are both infinite.

#### 4.1 Disconnected case

We assume in this section that  $(SS, dom)$  is disconnected and we write  $\mathbb{R} = \mathbb{R}' \times \mathbb{R}''$ . Let  $L \subseteq \mathbb{R}$ . If  $L$  is locally monitorable then, necessarily  $s\mathbb{R} \subseteq L$  or  $s\mathbb{R} \subseteq L^c$  for some prime  $s \in \mathbb{P} = \mathbb{P}(\mathbb{R}') \cup \mathbb{P}(\mathbb{R}'')$ . By symmetry we may assume  $s \in \mathbb{P}(\mathbb{R}')$  and  $s\mathbb{R} \subseteq L$ . As a consequence, there is no  $t \in \mathbb{P}(\mathbb{R}'')$  such  $t\mathbb{R} \subseteq L^c$ . On the other hand, if there is some prime  $t \in \mathbb{P}(\mathbb{R}'')$  such  $t\mathbb{R} \subseteq L$ , then  $L$  is locally monitorable for a trivial reason: For every prime trace  $u \in \mathbb{P}$  we either have  $u \in \mathbb{R}'$  or  $u \in \mathbb{R}''$ ; and by choosing either the prime  $s$  or  $t$  in the other component as  $u$  we satisfy the required condition for  $L$  to be locally monitorable.

Hence we are only interested in the case that there is no prime  $t \in \mathbb{R}''$  such that  $t\mathbb{R} \subseteq L$ . In this case we can reduce the problem whether  $L$  is locally monitorable to the component of  $\mathbb{R}'$  as follows: First, let us define languages of prime traces  $L_1 = \{u \in \mathbb{P}(\mathbb{R}') \mid u\mathbb{R} \subseteq L\}$  and  $L_2 = \{u \in \mathbb{P}(\mathbb{R}') \mid u\mathbb{R} \subseteq L^c\}$ . Note that if  $L$  is recognizable, then  $L_1, L_2$ , as well as  $L_1\mathbb{R}', L_2\mathbb{R}'$ , are recognizable too. Moreover, we can construct the corresponding automata.

**Theorem 4.** Let  $L \subseteq \mathbb{R} = \mathbb{R}' \times \mathbb{R}''$  and assume that there is some  $s \in \mathbb{P}(\mathbb{R}')$  such that  $s\mathbb{R} \subseteq L$  but there is no  $t \in \mathbb{P}(\mathbb{R}'')$  with  $t\mathbb{R} \subseteq L$ . Then  $L$  is locally monitorable if and only if the family  $\{L_1\mathbb{R}', L_2\mathbb{R}'\}$  is locally monitorable w.r.t.  $\mathbb{R}'$ .

*Proof.* First, let  $L$  be locally monitorable and  $s \in \mathbb{P}$  be a prime. Choose some prime  $t \in \mathbb{P}$  with  $s \leq t\mathbb{R}$  such that either  $t\mathbb{R} \subseteq L$  or  $t\mathbb{R} \subseteq L^c$ . We cannot have  $t \in R''$ , hence  $t \in \mathbb{P}(R')$ . Thus, either  $t \in L_1$  or  $t \in L_2$ . It follows that  $t\mathbb{R}' \subseteq L_1\mathbb{R}'$  or  $t\mathbb{R}' \subseteq L_2\mathbb{R}'$ , and hence  $\{L_1\mathbb{R}', L_2\mathbb{R}'\}$  is locally monitorable w.r.t.  $\mathbb{R}'$ .

For the other direction let  $\{L_1\mathbb{R}', L_2\mathbb{R}'\}$  be locally monitorable w.r.t.  $\mathbb{R}'$ . Then for every prime  $u \in \mathbb{P}(\mathbb{R}')$  there is some  $v \in \mathbb{P}(\mathbb{R}')$  such that  $u \leq v\mathbb{R}'$  such that either  $v\mathbb{R}' \subseteq L_1\mathbb{R}'$  or  $v\mathbb{R}' \subseteq L_2\mathbb{R}'$ . In particular, either  $v \in L_1$  or  $v \in L_2$ , since  $r \leq v$  with  $r \in L_i$  implies  $v \in L_i$ . By definition, either  $v\mathbb{R} \subseteq L$  or  $v\mathbb{R} \subseteq L^c$ . Thus,  $L$  is locally monitorable on all primes of  $\mathbb{R}'$ . Now, let  $u \in \mathbb{P}(\mathbb{R}'')$ . By assumption there is some  $s \in \mathbb{P}(\mathbb{R}')$  such that  $s\mathbb{R} \subseteq L$ . Since  $\mathbb{R} = \mathbb{R}' \times \mathbb{R}''$  we have  $u \leq s\mathbb{R}$ . Thus,  $L$  is locally monitorable.

## 4.2 Connected case

Recall that a distributed alphabet  $(SS, dom)$  is *connected* if it cannot be partitioned as  $SS = SS_1 \cup SS_2$  such that  $SS_1 \times SS_2 \subseteq I$  with  $SS_1 \neq \emptyset \neq SS_2$ . For connected  $(SS, dom)$  we obtain a nicer characterization of locally monitorable sets:

**Lemma 4.** *Let  $(\Sigma, dom)$  be connected. Then  $L$  is locally monitorable if and only if*

$$\forall s \in \mathbb{P} \exists s \leq t \in \mathbb{P} : t\mathbb{R} \subseteq L \vee t\mathbb{R} \subseteq L^c.$$

*Proof.* Let  $L$  be such that  $\forall s \in \mathbb{P} \exists t \in \mathbb{P} : s \leq t\mathbb{R} \subseteq L \vee s \leq t\mathbb{R} \subseteq L^c$ . We have to show that we can choose  $s$  to be a prefix of  $t$ . But this is clear: if  $s \leq t\mathbb{R}$ , then there is a prime  $p$  with  $s \leq p$  and  $t \leq p$ . The result follows because  $p\mathbb{R} \subseteq t\mathbb{R}$  in this case.

**Proposition 5.** *The following assertions are equivalent.*

1.  $(\Sigma, dom)$  is connected.
2. The family of locally monitorable sets is closed under finite union.
3. The family of locally monitorable sets is a Boolean algebra.

*Proof.* Since the locally monitorable property is symmetric for  $L, L^c$ , the last two items of the proposition are equivalent. Let  $(\Sigma, dom)$  be connected, we show that locally monitorable is preserved by taking finite unions. Let  $L$  and  $K$  be locally monitorable and consider  $s \in \mathbb{P}$ . If we find  $s \leq t \in \mathbb{P}$  and either  $t\mathbb{R} \subseteq L$  or  $t\mathbb{R} \subseteq K$ , we are done. Hence there is  $s \leq t \in \mathbb{P}$  and  $t\mathbb{R} \subseteq L^c$ . Now, we may assume that there is  $t \leq u \in \mathbb{P}$  and  $u\mathbb{R} \subseteq K^c$ . But then  $s \leq u$  and  $u\mathbb{R} \subseteq (L \cup K)^c$ .

Conversely, let  $a, b \in \Sigma$  be in different connected components of  $(\Sigma, dom)$  and let  $L$  = “no occurrence of  $a$ ” and  $K$  = “no occurrence of  $b$ ”. Both sets are locally monitorable, since they are prime-closed. However, for every prime  $s$  we have  $s \in L \cup K$  and  $s\mathbb{R} \cap (L \cup K)^c \neq \emptyset$ . This shows that  $L \cup K$  is not locally monitorable.

Again, for connected alphabets and a family of languages, we can make the condition to be locally monitorable more precise by using Lem. 4. Indeed, if  $(\Sigma, \text{dom})$  is connected, then a family  $\{L_1, \dots, L_n\}$  is locally monitorable if and only if

$$\forall s \in \mathbb{P} \exists s \leq t \in \mathbb{P} \exists 1 \leq i \leq n : t\mathbb{R} \subseteq L_i.$$

**Theorem 5.** *Let  $(\Sigma, \text{dom})$  be connected, and  $L_1, \dots, L_n$  be subsets of  $\mathbb{R}$  such that*

1.  $\mathbb{R} = L_1 \cup \dots \cup L_n$ .
2. Each  $L_k$  is a countable union of prime-closed sets.

Then the family  $\{L_1, \dots, L_n\}$  is locally monitorable.

*Proof.* We give the proof for  $n = 2$ , the one for  $n > 2$  is similar. Let  $L = L_1^c$  and  $K = L_2^c$ . Write  $L = \bigcap_{i \geq 0} U_i \mathbb{R}$  and  $K = \bigcap_{i \geq 0} V_i \mathbb{R}$  where all  $U_i, V_i \subseteq \mathbb{P}$ . Without restriction we have  $U_0 \mathbb{R} = V_0 \mathbb{R} = \mathbb{R}$ .

By contradiction, assume that  $\{L_1, L_2\}$  is not locally monitorable. This means that we can find some  $s \in \mathbb{P}$  such that for all  $t \in \mathbb{P}$  with  $\{s, t\}$  coherent it holds that  $t\mathbb{R} \cap L \neq \emptyset \neq t\mathbb{R} \cap K$ . Let  $p_0 = x_0 = q_0 = y_0 = s$ .

By induction let for some  $k \geq 1$  prime traces  $p_i, x_i, q_i$ , and  $y_i$  for all  $0 \leq i < k$  be defined such that  $U_i \ni p_i \leq x_i \leq y_i$ ,  $V_i \ni q_i \leq y_i$ , and  $y_{i-1} \leq x_i$ .

We define  $x_k, p_k$  as follows. Since  $s \leq y_{k-1} \in \mathbb{P}$  we have by assumption  $y_{k-1}\mathbb{R} \cap L \neq \emptyset$ , and thus we find  $y_{k-1} \leq x \in L$ . Thus, there is  $p_k \in U_k$  with  $p_k \leq x$ . The set  $\{y_{k-1}, p_k\}$  is coherent, hence there is common finite trace  $w$  with  $y_{k-1} \leq w$  and  $p_k \leq w$ . Since  $(\Sigma, dom)$  is connected, we find some prime  $x_k \in \mathbb{P}$  with  $w \leq x_k$ . The definition of  $y_k$  follows the same pattern. We have  $s \leq x_1 \leq y_1 \leq x_2 \cdots$  and  $x = \sqcup \{x_i \mid i \in \mathbb{N}\}$  exists. However,  $x \in \bigcap_{i \geq 0} U_i \mathbb{R} \cap \bigcap_{i \geq 0} V_i \mathbb{R}$ . Contradiction, because  $L \cap K = \emptyset$ .

*Remark 3.* Notice that the above proof still works if  $(SS, dom)$  has only two connected components. In the general case it is open whether the statement of Thm. 5 still holds.

## 5 Infinite traces

Prime-closed languages are prefix closed, so they always intersect. In particular, for any language  $L \subseteq \mathbb{R}$ , it can never happen that both  $L$  and  $L^c$  are countable unions of prime-closed sets (or equivalently, countable intersections of prime-open sets), as required by Thm. 5.

Thus, in order to define an trace analogue of  $G_\delta \cap F_\sigma$  we will restrict our attention to infinite traces where a (given) subset  $I$  of processes is active infinitely often and “sees” all other processes. In this way monitoring can be performed by processes in  $I$ . Another motivation for the new notion is due to the fact that in order to monitor a language we should be able to gather information into longer and longer prime prefixes.

For a finite trace  $t$  we write  $\max(t) \subseteq \Gamma$  if  $\text{dom}(a) \cap \Gamma \neq \emptyset$  for each  $a \in \max(t)$ .

**Definition 4.** Let  $\Gamma$  be a (non-empty) subset of  $Proc$ . A trace  $x$  is called  $\Gamma$ -infinite if

- Every process from  $\Gamma$  has infinitely many actions in  $x$ .
- $x$  can be written as  $x = x_0x_1 \cdots$  such that  $\max(x_n) \subseteq \Gamma$  for each  $n \geq 0$ .
- $\text{alph}(x)$  is connected.

The set of  $\Gamma$ -infinite traces is written as  $\mathbb{R}_\Gamma$ .

*Remark 4.* If  $\Gamma$  is a singleton, then for every trace  $x \in \mathbb{R}_\Gamma$ , both  $\text{alph}(x)$  and  $\text{alphinf}(x)$  are connected (and non-empty).

In the following everything is within  $\Gamma$ -infinite traces, for a fixed set  $\Gamma \subseteq Proc$ . In particular, the notion of closed and open are meant to be induced. The notion of locally monitorable is also relative to  $\mathbb{R}_\Gamma$ : a set  $L \subseteq \mathbb{R}_\Gamma$  is locally monitorable if  $\forall s \in \mathbb{P}(\mathbb{R}_\Gamma) \exists s \leq t \in \mathbb{P}(\mathbb{R}_\Gamma) : t\mathbb{R} \cap \mathbb{R}_\Gamma \subseteq L \vee t\mathbb{R} \cap \mathbb{R}_\Gamma \subseteq L^c$  (where  $L^c = \mathbb{R}_\Gamma \setminus L$ ).

**Definition 5.** Let  $\Gamma \subseteq Proc$  be a non-empty set of processes.

1. A set  $X \subseteq \mathbb{R}_\Gamma$  is prime- $G_\delta$  if it has the form  $X = \bigcap_{i \geq 0} U_i$  where all  $U_i$  are prime-open in  $\mathbb{R}_\Gamma$ . The family of prime- $G_\delta$ -sets is denoted  $\text{PG}_\delta$ .
2. A set  $X \subseteq \mathbb{R}_\Gamma$  is prime- $F_\sigma$  if its complement is prime- $G_\delta$ . The family of prime- $G_\delta$ -sets is denoted  $\text{PF}_\sigma$ .

*Example 3.* Let  $\Gamma = Proc = \{\alpha, \beta\}$  and  $SS = \{a, b, d\}$  with  $\text{dom}(a) = \{\alpha\}$ ,  $\text{dom}(b) = \{\beta\}$  and  $\text{dom}(d) = \{\alpha, \beta\}$ . Let  $L \subseteq \mathbb{R}_\Gamma$  contain all traces without the (trace) factor  $abd$ . Such traces are formed either by a trace from  $((a^* + b^*)d^+)^*(a^* + b^*)d^+$  followed by  $a^\omega b^\omega$ , or they belong to  $((a^* + b^*)d^+)^\omega$ . Clearly,  $L$  is prime-closed. The complement of  $L$  is in  $\text{PF}_\sigma$ , since  $L^c = \bigcup_{w \in SS^*, i, j > 0} X_{w, i, j}$  where  $X_{w, i, j}$  contains all traces from  $\mathbb{R}_\Gamma$  with prefix  $wa^i b^j d$ . Each  $X_{w, i, j}$  is prime-closed.

The next lemma generalizes the case of  $\omega$ -words. Note that we need the restriction to  $\mathbb{R}_\Gamma$  (or some similar restriction). As an example, consider  $SS = \{a, b\}$  with  $(a, b) \in I$ . The language  $L = a\mathbb{R}$  is prime-open. But its complement  $L^c = b^\infty$  cannot be written as countable intersection of prime-open sets in  $\mathbb{R}$ , since we cannot avoid occurrences of  $a$  in such sets.

**Lemma 5.** Prime-closed sets of  $\mathbb{R}_\Gamma$  are in  $\text{PG}_\delta$ .

*Proof.* Let  $L \subseteq \mathbb{R}_\Gamma$  be prime-closed. By definition, every  $\sqcup C \in \mathbb{R}_\Gamma$  where  $C$  is coherent and  $C \subseteq \mathbb{P}(L)$ , belongs to  $L$ . For  $K \subseteq \mathbb{P}$ ,  $\alpha \in \Gamma$  and  $k \in \mathbb{N}$  let

$$K_{\alpha, k} = \{p \in K \mid |p| \geq k, \alpha \in \text{dom}(\max(p))\}.$$

We claim that

$$L = \bigcap_{k \in \mathbb{N}, \alpha \in \Gamma} \mathbb{P}(L)_{\alpha, k} \mathbb{R}_\Gamma.$$

The inclusion from left to right follows from  $L \subseteq \mathbb{R}_\Gamma$  and the definition of  $\mathbb{R}_\Gamma$ . Let  $x \in \mathbb{R}_\Gamma$  be such that for every  $k \in \mathbb{N}$  and  $\alpha \in \Gamma$ , there is some  $p_{\alpha,k} \leq x$  with  $p_{\alpha,k} \in \mathbb{P}(L)_{\alpha,k}$ . By definition of  $\mathbb{R}_\Gamma$  and of  $\mathbb{P}(L)_{\alpha,k}$ , we have that  $x = \sqcup \{p_{\alpha,k} \mid k \in \mathbb{N}, \alpha \in \Gamma\}$ . Hence,  $x$  is of the form  $\sqcup C$  for  $C \subseteq \mathbb{P}(L)$  coherent, and thus in  $L$ .

**Theorem 6.** 1.  $\text{PG}_\delta \cap \text{PF}_\sigma$  is a Boolean algebra containing all prime-open and all prime-closed subsets of  $\mathbb{R}_\Gamma$ .  
2. All  $\text{PG}_\delta \cap \text{PF}_\sigma$  subsets of  $\mathbb{R}_\Gamma$  are locally monitorable.

*Proof.*  $\text{PG}_\delta$  is closed under union. Hence,  $\text{PG}_\delta \cap \text{PF}_\sigma$  is a Boolean algebra. It contains all prime-open and all prime-closed subsets of  $\mathbb{R}_\Gamma$  by Lem. 5.

The proof of the second claim follows along the same lines as the one of Thm. 5. Assume that  $\mathbb{R}_\Gamma \neq \emptyset$  and choose some connected subalphabet  $SS'$  of  $SS$  that contains for each  $\alpha \in \Gamma$  some letter  $a$  with  $\alpha \in \text{dom}(a)$ . The prime traces  $x_k, y_k$  can be chosen such that  $\max(x_k) \subseteq \Gamma$ ,  $\max(y_k) \subseteq \Gamma$ , and  $\text{alph}(x_k^{-1}y_k) = \text{alph}(y_{k-1}^{-1}x_k) = SS'$ . Thus,  $x = \sqcup_i x_i \in \mathbb{R}_\Gamma$ .

Asynchronous Büchi and Muller automata have been studied in [8,5]. McNaughton's theorem [13] stating the equivalence of non-deterministic Büchi and deterministic Muller automata over omega-word languages, extends to recognizable languages of infinite traces and asynchronous automata [5]. If we restrict to traces from  $\mathbb{R}_\Gamma$ , then the Büchi and Muller acceptance conditions are simpler:

**Definition 6.** Let  $\Gamma \subseteq \text{Proc}$  be a non-empty set of processes, and let  $\mathcal{A} = \langle (S_\alpha)_{\alpha \in \text{Proc}}, (\delta_a)_{a \in SS}, s^0 \rangle$  be an asynchronous automaton.

1. A Büchi acceptance condition is a set  $F \subseteq S_\Gamma$ .  
An infinite run  $s^0 = s_0, a_0, s_1, a_1, \dots$  of  $\mathcal{A}$  is accepting if for some  $f_\Gamma \in F$  and for every  $\alpha \in \Gamma$ , there are infinitely many  $n \geq 0$  with  $(s_n)_\alpha = f_\alpha$ .
2. A Muller acceptance condition is a set  $\mathcal{F} \subseteq \prod_{\alpha \in \Gamma} 2^{S_\alpha}$ .  
An infinite run  $s^0 = s_0, a_0, s_1, a_1, \dots$  of  $\mathcal{A}$  is accepting if for some  $T_\Gamma \in \mathcal{F}$  and for every  $\alpha \in \Gamma$ , the set of states from  $S_\alpha$  such that  $(s_n)_\alpha = f_\alpha$  for infinitely many  $n$ , is precisely  $T_\alpha$ .

The language  $\mathcal{L}(\mathcal{A})$  is the set of all traces from  $\mathbb{R}_\Gamma$  that have an accepting run. The next result is a generalization from  $\omega$ -word languages to  $\mathbb{R}_\Gamma$  trace languages:

**Theorem 7.** Let  $L \subseteq \mathbb{R}_\Gamma$  be recognizable. Then  $L$  is in  $\text{PG}_\delta$  if and only if  $L$  is accepted by a deterministic Büchi asynchronous automaton.

*Proof.* Assume first that  $L = \mathcal{L}(\mathcal{A})$ , where  $\mathcal{A}$  is a deterministic asynchronous Büchi automaton, and fix a final state  $f \in F$ . For  $n > 0, \alpha \in \Gamma$  we define  $K_{n,\alpha}^f$  as the set of all traces  $t \in \mathbb{P}$  with  $\alpha \in \text{dom}(\max(t))$  and such that in the run of  $\mathcal{A}$  on  $t$ , at least  $n$  letters on process  $\alpha$  are in state  $f_\alpha$ . It is easy to see that the set  $\bigcup_{f \in F} \bigcap_{\alpha \in \Gamma, n > 0} K_{n,\alpha}^f \mathbb{R}_\Gamma$  is precisely  $\mathcal{L}(\mathcal{A})$ . The remaining of the proof will show that  $\text{PG}_\delta$  is closed under finite union, thus  $\mathcal{L}(\mathcal{A}) \in \text{PG}_\delta$ .

For the converse let  $L = \bigcap_{n>0} U_n \subseteq \mathbb{R}_\Gamma$  be recognizable, with  $U_n$  prime-open in  $\mathbb{R}_\Gamma$ . We first define  $V_n = \bigcap_{k \leq n} U_k$ . It is not difficult to see that each  $V_n$  can be assumed to be of the form  $K_n \mathbb{R}_\Gamma$  with  $\max(t) \subseteq \Gamma$  for each  $t \in K_n$ . Let now  $K'_n \subseteq K_n$  consist of all elements of  $K_n$  that have no proper prefix in  $K_n$ . Let  $K = \bigcup_{n>0} K'_n X_n$ , where  $X_n$  is the set of traces  $t$  such that (1)  $\max(t) \subseteq \Gamma$ , (2)  $|t|_\alpha \geq n$  for each  $\alpha \in \Gamma$ , and (3) no proper prefix of  $t$  satisfies (1) and (2).

Let us first show that  $L = \{\sqcup C \mid C \subseteq K, C \text{ coherent}\}$ . The inclusion from left to right follows from  $L = \bigcap_{n>0} U_n = \bigcap_{n>0} K_n \mathbb{R}_\Gamma = \bigcap_{n>0} K'_n \mathbb{R}_\Gamma = \bigcap_{n>0} K'_n X_n \mathbb{R}_\Gamma$ . Conversely, let  $t = x_0 x_1 \dots$  with  $x_0 \dots x_n \in K$  for all  $n$ . Observe that we must have infinitely many  $n$  such that  $x_0 \dots x_m \in K_n$  for some  $m$ , since  $K'_n$  is prefix-free. Thus,  $t \in V_n$  for infinitely many  $n$  and  $t \in U_n$  for all  $n$ .

To conclude, we show that if  $L = \{\sqcup C \mid C \subseteq K, C \text{ coherent}\}$  for some  $K$ , and  $L \subseteq \mathbb{R}_\Gamma$  is recognizable, then  $L$  is the language of a deterministic asynchronous Büchi automaton. We assume as above that  $\max(t) \subseteq \Gamma$  for all  $t \in K$ . Since  $L$  is recognizable, there is some deterministic Muller automaton  $\mathcal{A}$  with acceptance condition  $\mathcal{F}$  and  $\mathcal{L}(\mathcal{A}) = L$ . We may also assume that on every finite trace  $t$  the states of processes from  $\text{dom}(\max(t))$  reached on  $t$  determine the states of all other processes. First we test for every  $T \in \mathcal{F}$  if there is some trace from  $\mathbb{R}_\Gamma$  accepted with  $T$ . Without restriction this is the case for all  $T \in \mathcal{F}$ . For each  $T$  we can determine a reachable state  $s(T) \in \prod_{\alpha \in \Gamma} T_\alpha$  and finite traces  $t_0(T), t(T)$  with  $\max(t_0(T), \max(t(T))) \subseteq \Gamma$  such that (1)  $t_0(T)$  leads from the initial state to  $s(T)$ , (2)  $t(T)$  is a loop on state  $s(T)$  and (3) the set of  $\alpha$ -states in the loop  $t(T)$  is precisely  $T_\alpha$ . In addition,  $t_0(T)$  is connected.

We claim that  $\mathcal{A}$  accepts  $L$  with the following (Büchi) condition: a trace is accepted if for some  $T \in \mathcal{F}$ , every state from  $T_\alpha$  occurs infinitely often, for every  $\alpha \in \Gamma$ . It is clear that all of  $L$  is accepted in this way by  $\mathcal{A}$ . Conversely, let  $x$  be an arbitrary trace with  $\max(x) \subseteq \Gamma$  and looping on state  $s(T)$ . We have  $t_0 t(T)^\omega \in L$ , so there is some  $n_0$  and  $u_0$  in  $K$  such that  $u_0 \leq t_0 t(T)^{n_0}$ . Since  $t_0 t(T)^{n_0} x t(T)^\omega \in L$  we find some  $n_1$  such that  $u_1 \leq t_0 t(T)^{n_0} x t(T)^{n_1}$  for some  $u_1 \in K$  with  $u_0 < u_1$ . In this way we can build a trace  $t$  from  $\mathbb{R}_\Gamma$ ,  $t = t_0 t(T)^{n_0} x t(T)^{n_1} x \dots$ , with  $t = \sqcup_{n \geq 0} u_n \in \{\sqcup C \mid C \subseteq K, C \text{ coherent}\}$  and such that for each  $\alpha \in \Gamma$ , the set of states from  $S_\alpha$  repeated infinitely often is a superset of  $T_\alpha$ . The claim follows since  $L = \{\sqcup C \mid C \subseteq K, C \text{ coherent}\}$ .

*Remark 5.* For the previous proof we do not need the connectedness assumption in the definition of  $\mathbb{R}_\Gamma$ . On the other hand, it is open whether without this assumption all  $\text{PG}_\delta \cap \text{PF}_\sigma$  sets are still locally monitorable.

## Conclusion

Our aim in this paper was to propose a reasonable notion of distributed monitoring for asynchronous systems. We argued that distributed monitors should have the same structure as the system that is monitored. We showed that properties over  $\Gamma$ -infinite traces that are deterministic and co-deterministic, are locally monitorable. It would be interesting to consider alternative restrictions to

$\Gamma$ -infinite traces, that capture some reasonable (partial) knowledge about the asynchronous system and for which  $\text{PG}_\delta \cap \text{PF}_\sigma$  sets are locally monitorable.

## References

1. A. Bauer, M. Leucker, and C. Schallhart. Monitoring of real-time properties. In *Proceedings of FSTTCS'06*, number 433 in LNCS, pages 260–272. Springer, 2006.
2. A. Bauer, M. Leucker, and C. Schallhart. Runtime verification for LTL and TLTL. *ACM Trans. Softw. Eng. Methodol.* 20(4), 20(4), 2011.
3. R. Chadha, A. P. Sistla, and M. Viswanathan. On the expressiveness and complexity of randomization in finite state monitors. *J. ACM*, 56(5), 2009.
4. V. Diekert and P. Gastin. Local safety and local liveness for distributed systems. In *Perspectives in Concurrency Theory*, pages 86–106. IARCS-Universities, 2009.
5. V. Diekert and A. Muscholl. Deterministic asynchronous automata for infinite traces. *Acta Informatica*, 31:379–397, 1994.
6. V. Diekert and G. Rozenberg, editors. *The Book of Traces*. World Scientific, 1995.
7. W. Ebinger and A. Muscholl. Logical definability on infinite traces. *Theoretical Computer Science*, 154(3):67–84, 1996.
8. P. Gastin and A. Petit. Infinite traces. In V. Diekert and G. Rozenberg, editors, *The Book of Traces*. World Scientific, 1995.
9. B. Genest, H. Gimbert, A. Muscholl, and I. Walukiewicz. Optimal Zielonka-type construction of deterministic asynchronous automata. In *Proceedings ICALP'10*, volume 6199 of LNCS. Springer, 2010.
10. K. Gondi, Y. Patel, and A. P. Sistla. Monitoring the full range of omega-regular properties of stochastic systems. In *Proceedings of VMCAI'09*, number 5403 in LNCS, pages 105–119. Springer, 2009.
11. R. M. Keller. Parallel program schemata and maximal parallelism I. Fundamental results. *Journal of the Association of Computing Machinery*, 20(3):514–537, 1973.
12. A. Mazurkiewicz. Concurrent program schemes and their interpretations. DAIMI Rep. PB 78, Aarhus University, Aarhus, 1977.
13. R. McNaughton. Testing and generating infinite sequences by a finite automaton. *Information & Control*, 9:521–530, 1966.
14. A. Pnueli. The temporal logic of programs. In *18th Symposium on Foundations of Computer Science*, pages 46–57, 1977.
15. A. Pnueli and A. Zaks. PSL model checking and run-time verification via testers. In *Formal Methods*, volume 4085 of LNCS, pages 573–586. Springer, 2006.
16. A. Stefanescu, J. Esparza, and A. Muscholl. Synthesis of distributed algorithms using asynchronous automata. In *CONCUR*, number 2761 in LNCS, pages 27–41, 2003.
17. W. Zielonka. Notes on finite asynchronous automata. *R.A.I.R.O. — Informatique Théorique et Applications*, 21:99–135, 1987.